Exponential contractivity of modified Euler schemes for SDEs with super-linearity

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Outline

- Motivations
- Contractivity under a mixed probability distance
- Contractivity under the L^1-Wasserstein distance
- Error bound between invariant probability measures

This talk is based on a joint work with Mateusz B. Majka and Jian Wang.

Motivations

- An SDE:

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t.$$

- Advantage of EM: simplest and succinctest;
- Drawback of EM: linear growth;
- Variants of EM (global dissipativity+synchrous coupling):
  - Backward EM: Higham et al.'02; dos Reis, et al.'23, $\cdots$;
  - Tamed EM: Hutzenthaler, et al.'12, Kumar et al.'22, Sabanis'16, $\cdots$;
  - Truncated EM: Mao'15; Li, et al.'24, $\cdots$;
  - Adaptive EM: Fang-Giles'20; Reisinger-Stockinger'22, $\cdots$

Motivations

Long-time analysis of EM for SDEs with partial dissipativity+ linear growth:

- Eberle & Majka'19: SDEs driven by BMs;
- Huang & Majka & Wang'22: SDEs with general noises;
- Liu & Majka & Monmarché'23: synchronous coupling+L^2-Wasserstein contraction;
- Schuh & Whalley'24: kinetic Langevin samplers+mixed coupling;
- Durmus, et al.'21: non-asymptotic bounds+weighted Wasserstein distance+discrete sticky coupling;
- $\cdots$

Modified Euler scheme

- SDE with additive noise:

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \mathrm{d}W_t.$$

- A modified Euler scheme:

$$X^{\delta}_{(n+1)\delta} = \pi^{(\delta)}(X^{\delta}_{n\delta}) + b^{(\delta)}\big(\pi^{(\delta)}(X^{\delta}_{n\delta})\big)\delta + \triangle W_{n\delta},$$

where

  - $|b^{(\delta)}(x) - b(x)| \to 0$ as $\delta \to 0$;
  - $\pi^{(\delta)} : \mathbb{R}^d \to \mathbb{R}^d$ s.t.

$$\big|\pi^{(\delta)}(x) - \pi^{(\delta)}(y)\big| \le |x - y|. \tag{1}$$

- The modified Euler scheme:

$$X^{\delta}_{(n+1)\delta} = \pi^{(\delta)}(X^{\delta}_{n\delta}) + b^{(\delta)}\big(\pi^{(\delta)}(X^{\delta}_{n\delta})\big)\delta + \delta^{\frac{1}{2}}\xi_{n+1}.$$

Technical assumptions

Assume that

(H₁) $\exists\ R, C_R, K_R \geq 1$, $\theta \in (0, \frac{1}{2})$, and $\delta_0 \in (0,1]$ s.t. for $\delta \in (0, \delta_0]$

$$
\begin{aligned}
&\left|b^{(\delta)}(\pi^{(\delta)}(x)) - b^{(\delta)}(\pi^{(\delta)}(y))\right| \\
&\leq \left(C_R \mathbf{I}_{\{x\in B_R\}\cap\{y\in B_R\}} + K_R \delta^{-\theta} \mathbf{I}_{\{x\in B_R^c\}\cup\{y\in B_R^c\}}\right)|x-y|;
\end{aligned}
$$

(H₂) for $R > 0$ above, $\exists\ K_R^* > 0$ s.t. for all $\delta \in (0, \delta_0]$, and $x, y \in \mathbb{R}^d$ with $x \in B_R^c$ or $y \in B_R^c$,

$$
\begin{aligned}
&\left\langle \pi^{(\delta)}(x) - \pi^{(\delta)}(y), b^{(\delta)}(\pi^{(\delta)}(x)) - b^{(\delta)}(\pi^{(\delta)}(y)) \right\rangle \\
&\leq -K_R^* \left|\pi^{(\delta)}(x) - \pi^{(\delta)}(y)\right|^2.
\end{aligned}
$$

Example: the classical Euler scheme

- Assume that

(B$_1$) $\exists$ an $L > 0$ s.t.

$$|b(x) - b(y)| \leq L|x - y|;$$

(B$_2$) $\exists$ $R > 0$ and $K_R > 0$ s.t. for all $x, y \in \mathbb{R}^d$ with $x \in B_R^c$ or $y \in B_R^c$,

$$\langle x - y, b(x) - b(y) \rangle \leq -K_R|x - y|^2.$$

- The Euler scheme:

$$X^{\delta}_{(n+1)\delta} = X^{\delta}_{n\delta} + b(X^{\delta}_{n\delta})\delta + \delta^{\frac{1}{2}}\xi_{n+1}.$$

- Take $\pi^{(\delta)}(x) = x$ and $b^{\delta}(x) = b(x)$.

Example: the tamed Euler scheme

- Assume that

($\mathbf{A}_1$) $\exists\ L_1, L_2 > 0$ and $l^* \geq 0$ s.t.

$$|b(x) - b(y)| \leq L_1\big(1 + |x|^{l^*} + |y|^{l^*}\big)|x - y|,$$
$$\Big|b(x)|y|^{l^*} - b(y)|x|^{l^*}\Big| \leq L_2\big(1 + |x|^{l^*} + |y|^{l^*} + |x|^{l^*}|y|^{l^*}\big)|x - y|;$$

($\mathbf{A}_2$) $\exists\ L_3, L_4, L_5 > 0$ and $R \geq 0$ s.t. for $x, y \in \mathbb{R}^d$ with $x \in B_R^c$ or $y \in B_R^c$,

$$\langle x - y, b(x) - b(y)\rangle \leq -L_3\big(1 + |x|^{l^*} + |y|^{l^*}\big)|x - y|^2,$$
$$\Big\langle x - y, b(x)|y|^{l^*} - b(y)|x|^{l^*}\Big\rangle$$
$$\leq \big(L_4\big(1 + |x|^{l^*} + |y|^{l^*}\big) - L_5|x|^{l^*}|y|^{l^*}\big)|x - y|^2.$$

- The tamed Euler scheme: $X^\delta_{(n+1)\delta} = X^\delta_{n\delta} + \frac{b(X^\delta_{n\delta})\delta}{1+\delta^\theta|X^\delta_{n\delta}|^{l^*}} + \delta^{\frac{1}{2}}\xi_{n+1}$.
- The double well potential: $b(x) := -\nabla U(x)$ with $U(x) = \frac{1}{4}|x|^4 - \frac{1}{2}|x|^2$.

Example: the truncated Euler scheme

- Assume that

(C_1) for any $r>0$, $\exists$ a strictly increasing and continuous function $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ s.t. for all $x,y\in B_r$,

$$|b(x)-b(y)|\le\varphi(r)|x-y|;$$

(C_2) $\exists\ R,K_R>0$ s.t. for $x,y\in\mathbb{R}^d$ with $x\in B_R^c$ or $y\in B_R^c$,

$$\langle x-y,b(x)-b(y)\rangle\le -K_R|x-y|^2.$$

- The truncation mapping: for $\theta\in(0,\frac{1}{2})$,

$$\pi^{(\delta)}(x):=\frac{1}{|x|}\big(|x|\wedge\varphi^{-1}(\delta^{-\theta})\big)x\mathbf{I}_{\{|x|>0\}}.$$

- The truncated Euler scheme:

$$X^\delta_{(n+1)\delta}=\pi^{(\delta)}(X^\delta_{n\delta})+b(\pi^{(\delta)}(X^\delta_{n\delta}))\delta+\delta^{\frac{1}{2}}\xi_{n+1}.$$

Contractivity under a mixed probability distance

Theorem

Assume (1), $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$. *Then, for any* $\delta \in (0, \delta_0^*]$, $\exists \lambda \in (0,1)$ *s.t.*

$$\mathbb{W}_\rho\big(\mathscr{L}_{X_{n\delta}^{\delta,x}}, \mathscr{L}_{X_{n\delta}^{\delta,y}}\big) \le \mathrm{e}^{-\lambda n\delta}\rho(x,y), \quad n \ge 0.$$

- The cost function $\rho(x,y) = a\mathbf{I}_{\{|x-y|>0\}} + f(|x-y|)$ with

$$f(r) = 1 - \mathrm{e}^{-c_* r} + c_* \mathrm{e}^{-2Rc_*} r, \quad r \ge 0.$$

- $\lambda \in (0,1)$ is explicit.
- $f(r) \simeq r$.

Refined basic coupling

- Set $\rho(x,z) := \frac{\nu_x(\mathrm{d}z)}{\nu(\mathrm{d}z)} = \frac{(2\pi)^{-\frac{d}{2}}\left(\mathrm{e}^{-\frac{|z|^2}{2}} \wedge \mathrm{e}^{-\frac{|z-x|^2}{2}}\right)}{(2\pi)^{-\frac{d}{2}}\mathrm{e}^{-\frac{|z|^2}{2}}} \in (0,1]$.
- $(x)_\kappa := \left(1 \wedge \frac{\kappa}{|x|}\right) x \mathbf{I}_{\{|x|\neq 0\}}$.
- Define the iteration: for $\widehat{Z}^\delta_{n\delta} := \widehat{X}^\delta_{n\delta} - \widehat{Y}^\delta_{n\delta}$ with $\widehat{X}^\delta_{n\delta} := \pi^{(\delta)}(X^\delta_{n\delta}) + b^{(\delta)}(\pi^{(\delta)}(X^\delta_{n\delta}))\delta$,

$$
\begin{cases}
X^\delta_{(n+1)\delta} = \widehat{X}^\delta_{n\delta} + \delta^{\frac{1}{2}}\xi_{n+1} \\
Y^\delta_{(n+1)\delta} = \widehat{Y}^\delta_{n\delta} + \delta^{\frac{1}{2}}\Big\{\Big(\xi_{n+1} + \delta^{-\frac{1}{2}}(\widehat{Z}^\delta_{n\delta})_\kappa\Big)\mathbf{I}_{\{U_{n+1}\leq \frac{1}{2}\rho(-\delta^{-\frac{1}{2}}(\widehat{Z}^\delta_{n\delta})_\kappa, \xi_{n+1})\}} \\
+\Big(\xi_{n+1} - \delta^{-\frac{1}{2}}(\widehat{Z}^\delta_{n\delta})_\kappa\Big) \\
\times \mathbf{I}_{\{\frac{1}{2}\rho(-\delta^{-\frac{1}{2}}(\widehat{Z}^\delta_{n\delta})_\kappa,\xi_{n+1})\leq U_{n+1}\leq \frac{1}{2}(\rho(-\delta^{-\frac{1}{2}}(\widehat{Z}^\delta_{n\delta})_\kappa,\xi_{n+1})+\rho(\delta^{-\frac{1}{2}}(\widehat{Z}^\delta_{n\delta})_\kappa,\xi_{n+1}))\}} \\
+\xi_{n+1}\mathbf{I}_{\{\frac{1}{2}(\rho(-\delta^{-\frac{1}{2}}(\widehat{Z}^\delta_{n\delta})_\kappa,\xi_{n+1})+\rho(\delta^{-\frac{1}{2}}(\widehat{Z}^\delta_{n\delta})_\kappa,\xi_{n+1}))\leq U_{n+1}\leq 1\}}\Big\},
\end{cases}
$$

A key lemma

Lemma

Assume (1), $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$. *Then, for any* $\delta \in (0, \delta_0^*]$,

$$\overline{\mathbb{E}}\big(|X^\delta_{n\delta} - Y^\delta_{n\delta}|\big|(X^\delta_{(n-1)\delta}, Y^\delta_{(n-1)\delta})\big) - |X^\delta_{(n-1)\delta} - Y^\delta_{(n-1)\delta}|$$
$$\leq \Big(C_R\delta\mathbf{I}_{\{|X^\delta_{(n-1)\delta} - Y^\delta_{(n-1)\delta}|\leq 2R\}}$$
$$-\frac{1}{2}K_R^*\delta\mathbf{I}_{\{|X^\delta_{(n-1)\delta} - Y^\delta_{(n-1)\delta}|> 2R\}}\Big)|X^\delta_{(n-1)\delta} - Y^\delta_{(n-1)\delta}|.$$

Moreover, in case of $|X^\delta_{(n-1)} - Y^\delta_{(n-1)\delta}| \leq \kappa/(1 + C_R)$,

$$\overline{\mathbb{E}}\big(\mathbf{I}_{\{X^\delta_{n\delta} = Y^\delta_{n\delta}\}}\big|(X^\delta_{(n-1)\delta}, Y^\delta_{(n-1)\delta})\big) \geq \frac{1}{2}J\big(\kappa\delta^{-\frac{1}{2}}\big)$$

Remark: $\kappa = \infty \Rightarrow \frac{1}{2}J(\delta^{-\frac{1}{2}}(1 + C_R)|x - y|)$; $J(r) = \inf_{|u|\leq r}\nu_u(\mathbb{R}^d)$.

Contractivity under the L^1-Wasserstein distance

- Assume that

$(\mathbf{H}_1')$ $\exists\, R, C_R, K_R \ge 1$, $\theta \in (0, \frac{1}{2})$, and $\delta_0 \in (0,1]$ s.t. for any $\delta \in (0, \delta_0]$,

$$\left|b^{(\delta)}(x) - b^{(\delta)}(y)\right| \le \Big(C_R \mathbf{I}_{\{x\in B_R\}\cap\{y\in B_R\}} + K_R \delta^{-\theta} \mathbf{I}_{\{x\in B_R^c\}\cup\{y\in B_R^c\}}\Big)|x-y|;$$

$(\mathbf{H}_2')$ $\exists\, K_R^* > 0$ s.t. for all $\delta \in (0, \delta_0]$, and $x, y \in \mathbb{R}^d$ with $x \in B_R^c$ or $y \in B_R^c$,

$$\langle x-y, b^{(\delta)}(x) - b^{(\delta)}(y)\rangle \le -K_R^* |x-y|^2.$$

- A modified Euler scheme:

$$X^\delta_{(n+1)\delta} = X^\delta_{n\delta} + b^{(\delta)}\big(X^\delta_{n\delta}\big)\delta + \delta^{\frac{1}{2}}\xi_{n+1}.$$

Contractivity under the L^1-Wasserstein distance

Theorem

Under $(\mathbf{H}_1')$ *and* $(\mathbf{H}_2')$, $\exists\ C > 0$ *and* $\lambda \in (0,1)$ *s.t. for all* $\delta \in (0, \delta_0^\star]$,

$$\mathbb{W}_1\big(\mathscr{L}_{X_{n\delta}^{\delta,x}}, \mathscr{L}_{X_{n\delta}^{\delta,y}}\big) \le C e^{-\lambda n\delta}|x-y|.$$

- An essential ingredient: as $\delta \to 0$,

$$\Big|\big(\pi^\delta(x) + b^{(\delta)}(\pi^\delta(x))\delta - (\pi^\delta(y) + b^{(\delta)}(\pi^\delta(y))\delta\big) - (x-y)\Big| \to 0.$$

- The truncation mapping:

$$\pi^{(\delta)}(x) := \frac{1}{|x|}\big(|x| \wedge \varphi^{-1}(\delta^{-\theta})\big)x\mathbf{I}_{\{|x|>0\}}.$$

Coupling by reflection

- Consider the recursion: for any $\delta, \kappa^* > 0$ and integer $n \geq 0$,

$$\begin{cases} X^\delta_{(n+1)\delta} = \widehat{X}^\delta_{n\delta} + \delta^{\frac{1}{2}}\xi_{n+1} \\ Y^\delta_{(n+1)\delta} = \widehat{Y}^\delta_{n\delta} + \delta^{\frac{1}{2}}\Big\{\big(\xi_{n+1} + \delta^{-\frac{1}{2}}(\widehat{Z}^\delta_n)_{\kappa^*}\big)\mathbf{I}_{\{U_{n+1}\leq\rho(-\delta^{-\frac{1}{2}}(\widehat{Z}^\delta_n)_{\kappa^*},\xi_{n+1})\}} \\ \qquad\qquad + \Pi(\widehat{X}^\delta_n - \widehat{Y}^\delta_n)\xi_{n+1}\mathbf{I}_{\{(\rho(-\delta^{-\frac{1}{2}}(\widehat{Z}^\delta_n)_{\kappa^*},\xi_{n+1})\leq U_{n+1}\leq 1\}}\Big\}. \end{cases}$$

- Intuition:
  - Basic coupling: $\frac{1}{2}\nu_{-\delta^{-\frac{1}{2}}\widehat{z}^\delta}(\mathrm{d}z)$.
  - Coupling by reflection: the remaining mass.

- Assume that

($\mathbf{A}_1$) $\exists\ L_1, L_2 > 0$ and $l^* \geq 0$ s.t.

$$|b(x) - b(y)| \leq L_1\big(1 + |x|^{l^*} + |y|^{l^*}\big)|x - y|, \tag{2}$$

$$\Big|b(x)|y|^{l^*} - b(y)|x|^{l^*}\Big| \leq L_2\big(1 + |x|^{l^*} + |y|^{l^*} + |x|^{l^*}|y|^{l^*}\big)|x - y|;$$

($\mathbf{A}_2$) $\exists\ L_3, L_4, L_5 > 0$ and $R \geq 0$ s.t. for $x, y \in \mathbb{R}^d$ with $x \in B_R^c$ or $y \in B_R^c$,

$$\langle x - y, b(x) - b(y)\rangle \leq -L_3\big(1 + |x|^{l^*} + |y|^{l^*}\big)|x - y|^2, \tag{3}$$

$$\Big\langle x - y, b(x)|y|^{l^*} - b(y)|x|^{l^*}\Big\rangle$$

$$\leq \big(L_4\big(1 + |x|^{l^*} + |y|^{l^*}\big) - L_5|x|^{l^*}|y|^{l^*}\big)|x - y|^2.$$

- The tamed Euler scheme:

$$X^\delta_{(n+1)\delta} = X^\delta_{n\delta} + \frac{b(X^\delta_{n\delta})\delta}{1 + \delta^\theta|X^\delta_{n\delta}|^{l^*}} + \delta^{\frac{1}{2}}\xi_{n+1}.$$

Error bound between invariant probability measures

Theorem

Under $(\mathbf{A}_1)$ *and* $(\mathbf{A}_2)$, $\exists\, C > 0$ *s.t. for all* $\delta \in (0, \delta_0^{\star\star\star}]$,

$$\mathbb{W}_1(\pi, \pi^\delta) \le C\delta^\theta,$$

where $\theta \in (0, 1/2)$, $\pi \in \mathscr{P}_1(\mathbb{R}^d)$ *and* $\pi^\delta \in \mathscr{P}_1(\mathbb{R}^d)$ *stand respectively for the unique invariant probability measures of* $(X_t)_{t\ge0}$ *and* $(X_{n\delta}^\delta)_{n\ge0}$.

- Via the triangle inequality,

$$\mathbb{W}_1(\pi, \pi^\delta) \le \mathbb{W}_1\Big(\mathscr{L}_{X_{n\delta}^\pi}, \mathscr{L}_{X_{n\delta}^{\mathbf{0}}}\Big) + \mathbb{W}_1\Big(\mathscr{L}_{X_{n\delta}^{\mathbf{0}}}, \mathscr{L}_{X_{n\delta}^{\delta,\mathbf{0}}}\Big) + \mathbb{W}_1\Big(\mathscr{L}_{X_{n\delta}^{\delta,\mathbf{0}}}, \mathscr{L}_{X_{n\delta}^{\delta,\pi^\delta}}\Big).$$

Asymptotic coupling by reflection (Wang'15)

Consider the coupled SDE: for any $\varepsilon, t > 0$,

$$\begin{cases} \mathrm{d}\overline{X}_t = b(\overline{X}_t)\,\mathrm{d}t + h_\varepsilon(|\overline{Z}_t|)\mathrm{d}\overline{W}_t + \left(1 - h_\varepsilon(|\overline{Z}_t|)^2\right)^{\frac{1}{2}}\,\mathrm{d}\overline{B}_t, \\ \mathrm{d}\overline{Y}_t = b^{(\delta)}(\overline{Y}_{\lfloor t/\delta \rfloor \delta})\,\mathrm{d}t + \Pi(\overline{Z}_t) h_\varepsilon(|\overline{Z}_t|)\mathrm{d}\overline{W}_t + \left(1 - h_\varepsilon(|\overline{Z}_t|)^2\right)^{\frac{1}{2}}\,\mathrm{d}\overline{B}_t, \end{cases}$$

where

- $\overline{Z}_t := \overline{X}_t - \overline{Y}_t$;
- $(\overline{W}_t)_{t\geq 0}$ and $(\overline{B}_t)_{t\geq 0}$: mutually independent Brownian motions;
- The truncation function:

$$h_\varepsilon(r) = \begin{cases} 0, & 0 \leq r \leq \varepsilon, \\ 1 - \mathrm{e}^{-\frac{r-\varepsilon}{2\varepsilon - r}}, & r \in (\varepsilon, 2\varepsilon), \\ 1, & r \geq 2\varepsilon. \end{cases}$$

Asymptotic coupling by reflection

Proposition

Under (2) *and* (3), $\exists\, C_*, \lambda_* > 0$ *s.t.*

$$\mathbb{W}_1\left(\mathscr{L}_{X^{\mathbf{0}}_{n\delta}}, \mathscr{L}_{X^{\delta,\mathbf{0}}_{n\delta}}\right) \le C_* \int_0^{n\delta} \mathrm{e}^{-\lambda_*(n\delta-s)}\mathbb{E}\Big|b(\overline{Y}_s) - b^{(\delta)}(\overline{Y}_{\lfloor s/\delta\rfloor\delta})\Big|\,\mathrm{d}s.$$

Further remarks

- SDEs driven by stable noises:

$$\mathbb{W}_\rho\big(\mathscr{L}_{X_{n\delta}^{\delta,x}},\mathscr{L}_{X_{n\delta}^{\delta,y}}\big)\le \mathrm{e}^{-\lambda_\delta n\delta}\rho(x,y),\quad n\ge 0.$$

  and

$$\mathbb{W}_1\big(\mathscr{L}_{X_{n\delta}^{\delta,x}},\mathscr{L}_{X_{n\delta}^{\delta,y}}\big)\le C\mathrm{e}^{-\lambda_\delta n\delta}|x-y|.$$

- Interacting particle systems.
- Kinetic Langevin samplers.
- More applications on sampling.